1

Sequence and Series of Real Numbers

1.1 Sequence of Real Numbers

Suppose for each positive integer n, we are given a real number a_n. Then, the list of numbers,

$$a_1, a_2, \ldots, a_n, \ldots,$$

is called a *sequence*, and this *ordered list* is usually written as

$$(a_1, a_2, \ldots, \ldots) \quad \text{or} \quad (a_n) \quad \text{or} \quad \{a_n\}.$$

More precisely, we define a sequence as follows:

Definition 1.1 A **sequence** of real numbers is a function from the set $\mathbb{N}$ of natural numbers to the set $\mathbb{R}$ of real numbers. If $f : \mathbb{N} \to \mathbb{R}$ is a sequence, and if $a_n = f(n)$ for $n \in \mathbb{N}$, then we write the sequence f as (a_n) or $(a_1, a_2, \ldots)$.

A sequence of real numbers is also called a *real sequence*. □

Remark 1.1 (a) It is to be born in mind that a sequence $(a_1, a_2, \ldots, \ldots)$ is different from the set $\{a_n : n \in \mathbb{N}\}$. For instance, a number may be repeated in a sequence (a_n), but it need not be written repeatedly in the set $\{a_n : n \in \mathbb{N}\}$. As an example, $(1, 1/2, 1, 1/3, \ldots, 1, 1/n, \ldots)$ is a sequence (a_n) with $a_{2n-1} = 1$ and $a_{2n} = 1/(n+1)$ for each $n \in \mathbb{N}$, where as the set $\{a_n : n \in \mathbb{N}\}$ is same as the set $\{1/n : n \in \mathbb{N}\}$.

(b) Instead of sequence of real numbers, we can also talk about a sequence of elements from any nonempty set S, such as sequence of sets, sequence of functions and so on. Thus, given a nonempty set S, a sequence in S is a function $f : \mathbb{N} \to S$. For example, for each $n \in \mathbb{N}$, consider the set $A_n = \{j \in \mathbb{N} : j \leq n\}$. Then we obtain a sequence of subsets of $\mathbb{N}$, namely, $(A_1, A_2, \ldots)$.

In this chapter, we shall consider only sequence of real numbers. In some of the later chapters we shall consider sequences of functions as well. ⧫

EXAMPLE 1.1 **(i)** (a_n) with $a_n = 1$ for all $n \in \mathbb{N}$ – a constant sequence with value 1 throughout.

(ii) (a_n) with $a_n = n$ for all $n \in \mathbb{N}$.

(iii) (a_n) with $a_n = 1/n$ for all $n \in \mathbb{N}$.

(iv) (a_n) with $a_n = n/(n+1)$ for all $n \in \mathbb{N}$.

(v) (a_n) with $a_n = (-1)^n$ for all $n \in \mathbb{N}$ – the sequence takes values 1 and -1 alternately. $\square$

Question: Consider a sequence $(a_1, a_2, \ldots)$. Is $(a_2, a_3, \ldots)$ also a sequence? Why?

1.1.1 Convergence and divergence

A fundamental concept in mathematics is that of *convergence*. We consider convergence of sequences.

Consider the sequences listed in Example 1.1 and observe the way a_n vary as n becomes larger and larger:

(i) $a_n = 1$: every term of the sequence is same.

(ii) $a_n = n$: the terms becomes larger and larger.

(iii) $a_n = 1/n$: the terms come closer to 0 as n becomes larger and larger.

(iv) $a_n = n/(n+1)$: the terms come closer to 1 as n becomes larger and larger.

(v) $a_n = (-1)^n$: the terms of the sequence oscillates with values -1 and 1, and does not come closer to any number as n becomes larger and larger.

Now, we make precise the statement "a_n comes closer to a number a" as n becomes larger and larger.

Definition 1.2 A sequence (a_n) in $\mathbb{R}$ is said to **converge** to a real number a if for every $\varepsilon > 0$, there exists positive integer N (in general depending on ε) such that

$$|a_n - a| < \varepsilon \qquad \forall\, n \geq N,$$

and in that case, the number a is called a **limit** of the sequence (a_n), and (a_n) is called a **convergent sequence**. $\square$

Remark 1.2 (a) Note that, different ε can result in different N, i.e., the number N may vary as ε varies. We shall illustrate this in Example 1.2.

(b) In Definition 1.2, the relation $|a_n - a| < \varepsilon$ can be replaced by $|a_n - a| < c_0\varepsilon$ for any $c_0 > 0$ which is independent of n (Why?). Thus, given a (a_n) in $\mathbb{R}$, if we are able to identify a number $a \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ satisfying $|a_n - a| < c_0\,\varepsilon$ for all $n \geq N$ for some constant $c_0 > 0$, then (a_n) converges to a. ♦

NOTATION: If (a_n) converges to a, then we write

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a \quad \text{as} \quad n \to \infty$$

or simply as $a_n \to a$.

Note that

$$|a_n - a| < \varepsilon \qquad \forall\, n \geq N$$

if and only if

$$a - \varepsilon < a_n < a + \varepsilon \qquad \forall\; n \geq N.$$

Thus, $\lim_{n\to\infty} a_n = a$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n \in (a - \varepsilon,\, a + \varepsilon) \quad \forall\, n \geq N.$$

Thus, $a_n \to a$ if and only if for every $\varepsilon > 0$, a_n belongs to the open interval $(a-\varepsilon,\, a+\varepsilon)$ for all n after some finite stage, and this finite stage may vary according as ε varies.

Remark 1.3 Suppose (a_n) is a sequence and $a \in \mathbb{R}$. Then to show that (a_n) does not converge to a, we should be able to find an $\varepsilon > 0$ such that infinitely may a_n's are outside the interval $(a - \varepsilon, a + \varepsilon)$. ♦

Exercise 1.1 Show that a sequence (a_n) converges to a if and only if for every open interval I containing a, there exits $N \in \mathbb{N}$ such that $a_n \in I$ for all $n \geq N$. ◄

Before further discussion, let us observe an important property.

Theorem 1.1 *Limit of a convergent sequence is unique. That, is if $a_n \to a$ and $a_n \to a'$ as $n \to \infty$, then $a = a'$.*

Proof. Suppose $a_n \to a$ and $a_n \to a'$ as $n \to \infty$, and suppose that $a' \neq a$. Now, for $\varepsilon > 0$, suppose $N_1, N_2 \in \mathbb{N}$ be such that

$$a_n \in (a - \varepsilon,\, a + \varepsilon) \quad \forall\, n \geq N_1, \quad a_n \in (a' - \varepsilon,\, a' + \varepsilon) \quad \forall\, n \geq N_2.$$

In particular,

$$a_n \in (a - \varepsilon,\, a + \varepsilon), \quad a_n \in (a' - \varepsilon,\, a' + \varepsilon) \quad \forall\, n \geq N := \max\{N_1, N_2\}.$$

If we take $\varepsilon < |a - a'|/2$, then we see that $(a - \varepsilon,\, a + \varepsilon)$ and $(a' - \varepsilon,\, a' + \varepsilon)$ are disjoint intervals. Thus the above observation leads to a contradiction. ∎

An alternate proof. Note that

$$|a - a'| = |(a - a_n) + (a_n - a')| \leq |a - a_n| + |a_n - a'|.$$

Now, for $\varepsilon > 0$, let $N_1, N_2 \in \mathbb{N}$ be such that

$$|a - a_n| < \varepsilon/2 \quad \text{for all} \quad n \geq N_1, \quad |a' - a_n| < \varepsilon/2 \quad \forall\, n \geq N_2.$$

Then it follows that

$$|a - a'| \leq |a - a_n| + |a_n - a'| < \varepsilon \qquad \forall\, n \geq N := \max\{N_1, N_2\}.$$

Since this is true for all $\varepsilon > 0$, it follows that $a' = a$. ∎

EXAMPLE 1.2 The sequences $(1/n)$, $((-1)^n/n)$, $(1-\frac{1}{n})$ are convergent with limit 0, 0, 1 respectively:

For the sake of illustrating how to use the definition to justify the above statement, let us provide the details of the proofs:

(i) Let $a_n = 1/n$ for all $n \in \mathbb{N}$, and let $\varepsilon > 0$ be given. We have to identify an $N \in \mathbb{N}$ such that $1/n < \varepsilon$ for all $n \geq N$. Note that

$$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}.$$

Thus, if we take $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$, then we have

$$|a_n - 0| = \frac{1}{n} < \varepsilon \qquad \forall\, n \geq N.$$

Hence, $(1/n)$ converges to 0.

Here $\lceil x \rceil$ denotes the integer part of x.

(ii) Next, let $a_n = (-1)^n/n$ for all $n \in \mathbb{N}$. Since $|a_n| = 1/n$ for all $n \in \mathbb{N}$, in this case also, we see that

$$|a_n - 0| < \varepsilon \qquad \forall\, n \geq N := \left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Hence, $((-1)^n/n)$ converges to 0.

(iii) Now, let $a_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Since, $|a_n - 1| = 1/n$ for all $n \in \mathbb{N}$, we have

$$|a_n - 1| < \varepsilon \qquad \forall\, n \geq N := \left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Hence, $(1 - 1/n)$ converges to 0. $\square$

EXAMPLE 1.3 Every constant sequence is convergent to the constant term in the sequence.

To see this, let $a_n = a$ for all $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, we have

$$|a_n - a| = 0 < \varepsilon \qquad \forall\, n \geq N := 1.$$

Thus, (a_n) converges to a. $\square$

EXAMPLE 1.4 For a given $k \in \mathbb{N}$, let Let $a_n = 1/n^{1/k}$ for all $n \in \mathbb{N}$. Then $a_n \to 0$ as $n \to \infty$.

To see this, first let $\varepsilon > 0$ be given. Note that

$$\frac{1}{n^{1/k}} < \varepsilon \iff n > \frac{1}{\varepsilon^k}.$$

Hence,

$$\frac{1}{n^{1/k}} < \varepsilon \qquad \forall\, n \geq N := \left\lceil \frac{1}{\varepsilon^k} \right\rceil + 1.$$

Thus, $1/n^{1/k} \to 0$. □

Exercise 1.2 Corresponding to a sequence (a_n) and $k \in \mathbb{N}$, let (b_n) be defined by $b_n = a_{k+n}$ for all $n \in \mathbb{N}$. Show that, for $a \in \mathbb{R}$, $a_n \to a$ if and only if $b_n \to a$. ◀

Definition 1.3 A sequence (a_n) is said to be **eventually constant** if there exists $k \in \mathbb{N}$ such that $a_{k+n} = a_k$ for all $n \geq 1$. □

Exercise 1.3 Show that every eventually constant sequence converges. ◀

Exercise 1.4 Prove the following.

1. Let $b \geq 0$ such that $b < \varepsilon$ for all $\varepsilon > 0$. Then $b = 0$.
2. Let $a_n \geq 0$ for all $n \in \mathbb{N}$ such that $a_n \to a$. Then $a \geq 0$.

◀

The following theorem can be easily proved.

Theorem 1.2 *Suppose $a_n \to a$, $b_n \to b$ as $n \to \infty$. Then we have the following :*

(i) $a_n + b_n \to a + b$ *as* $n \to \infty$,

(ii) *For every real number* c, $ca_n \to c\,a$ *as* $n \to \infty$

(iii) *If* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.

(iv) **(Sandwitch theorem)** *If* $a_n \leq c_n \leq b_n$ *for all* $n \in \mathbb{N}$, *and if* $a = b$, *then* $c_n \to a$ *as* $n \to \infty$.

Exercise 1.5 Supply proof for Theorem 1.2. ◀

Exercise 1.6 If $a_n \to a$ and there exists $b \in \mathbb{R}$ such that $a_n \geq b$ for all $n \in \mathbb{N}$, then show that $a \geq b$. ◀

Exercise 1.7 If $a_n \to a$ and $a \neq 0$, then show that there exists $k \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq k$. ◀

Exercise 1.8 Consider the sequence (a_n) with $a_n = \left(1 + \frac{1}{n}\right)^{1/n}$, $n \in \mathbb{N}$. Then show that $\lim_{n\to\infty} a_n = 1$.

[Hint: Observe that $1 \leq a_n \leq (1 + 1/n)$ for all $n \in \mathbb{N}$.] ◀

Exercise 1.9 Consider the sequence (a_n) with $a_n = \dfrac{1}{n^k}$, $n \in \mathbb{N}$. Then show that for any given $k \in \mathbb{N}$, $\lim_{n\to\infty} a_n = 0$.

[Hint: Observe that $1 \le a_n \le 1/n$ for all $n \in \mathbb{N}$.] ◀

Remark 1.4 In Theorem 1.2 (c) and (d), instead of assuming the inequalities for all $n \in \mathbb{N}$, we can assume them to hold for all $n \ge n_0$ for some $n_0 \in \mathbb{N}$ (Why?). ⧫

Definition 1.4 A sequence which does not converge is called a **divergent sequence**. □

Definition 1.5 (i) If (a_n) is such that for every $M > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n > M \qquad \forall\, n \ge N,$$

then we say that (a_n) **diverges to** $+\infty$.

(ii) If (a_n) is such that for every $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n < -M$ for all $n \ge N$, then we say that (a_n) **diverges to** $-\infty$. □

Definition 1.6 If (a_n) is such that $a_n a_{n+1} < 0$ for every $n \in \mathbb{N}$, that is a_n changes sign alternately, then we say that (a_n) is an **alternating** sequence . □

An alternating sequence converge or diverge. For example, (*Verify that*) the sequence $((-1)^n)$ diverges, whereas $((-1)^n/n)$ converges to 0.

Definition 1.7 A sequence (a_n) is said to be **bounded above** if there exists a real number M such that $a_n \le M$ for all $n \in \mathbb{N}$; and the sequence (a_n) is said to be **bounded below** if there exists a real number M' such that $a_n \ge M'$ for all $n \in \mathbb{N}$. A sequence which is bound above and bounded below is said to be a **bounded** sequence. □

Exercise 1.10 Show that a sequence (a_n) is bounded if and only if there exists $M > 0$ such that $|a_n| \le M$ for all $n \in \mathbb{N}$. ◀

Exercise 1.11 Prove the following.

1. If (a_n) is not bounded above, then there exists a strictly increasing sequence (k_n) of natural numbers such that $a_{k_n} \to +\infty$ as $n \to \infty$.
2. If (a_n) is not bounded below, then there exists a strictly increasing sequence (k_n) of natural numbers such that $a_{k_n} \to -\infty$ as $n \to \infty$.

◀

Theorem 1.3 *Every convergent sequence is bounded. The converse is not true.*

Proof. Suppose $a_n \to a$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| \leq 1$ for all $n \geq N$. Hence

$$|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| \leq 1 + |a| \qquad \forall\, n \geq N.$$

Thus, if take $K = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|\}$, then we have

$$|a_n| \leq \max\{1 + |a|,\, K\} \qquad \forall\, n \in \mathbb{N}.$$

To see that the converse of the theorem is not true, consider the sequence $((-1)^n)$. It is a bounded sequence, but not convergent. ∎

The above theorem can be used to show that certain sequence is not convergent, as in the following example.

EXAMPLE 1.5 For $n \in \mathbb{N}$, let

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

Then (a_n) diverges: To see this, observe that

$$\begin{aligned} a_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \\ &\quad \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \\ &\geq 1 + \frac{n}{2}. \end{aligned}$$

Hence, (a_n) is not a bounded sequence, so that it diverges. □

Using Theorem 1.3, the following result can be deduced

Theorem 1.4 *Suppose $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then we have the following.*

(i) $a_n b_n \to ab$ as $n \to \infty$.

(ii) If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $a_n/b_n \to a/b$ as $n \to \infty$.

Exercise 1.12 Prove Theorem 1.4. ◀

Exercise 1.13 If (a_n) converges to a and $a \neq 0$, then show that there exists $k \in \mathbb{N}$ such that $|a_n| \geq |a|/2$ for all $n \geq k$, and $(1/a_{n+k})$ converges to $1/a$. ◀

1.1.2 Monotonic sequences

We can infer the convergence or divergence of a sequence in certain cases by observing the way the terms of the sequence varies.

Definition 1.8 Consider a sequence (a_n).

(i) (a_n) is said to be **monotonically increasing** if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.

(ii) (a_n) is said to be **monotonically decreasing** if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$.

If strict inequality occur in (i) and (ii), then we say that the sequence is **strictly increasing** and **strictly decreasing**, respectively.

Also, a monotonically increasing (respectively, a monotonically decreasing) sequence is also called an increasing (respectively, a decreasing) sequence. □

We shall make use of some important properties of the set $\mathbb{R}$ of real numbers.

Definition 1.9 Let S be a subset of $\mathbb{R}$. Then

(i) S is said to be **bounded above** if there exists $b \in \mathbb{R}$ such that $x \le b$ for all $x \in S$, and in that case b is called an **upper bound** of S;

(ii) S is said to be **bounded below** if there exists $a \in \mathbb{R}$ such that $a \le x$ for all $x \in S$, and in that case a is called a **lower bound** of S. □

Definition 1.10 Let S be a subset of $\mathbb{R}$.

(i) A number $b_0 \in \mathbb{R}$ is called a **least upper bound** (lub) or **supremum** of $S \subseteq \mathbb{R}$ if b_0 is an upper bound of S and for any upper bound b of S, $b_0 \le b$.

(ii) A number $a_0 \in \mathbb{R}$ is called a **greatest lower bound** (glb) or **infimum** of $S \subseteq \mathbb{R}$ if a_0 is a lower bound of S and for any lower bound a of S, $a \le a_0$. □

Thus, we have the following:

- $b_0 \in \mathbb{R}$ is supremum of $S \subseteq \mathbb{R}$ if and only if b_0 is an upper bound of S and if $\beta < b_0$, then β is not an upper bound of S, i.e., there exists $x \in S$ such that $\beta < x$.

- $a_0 \in \mathbb{R}$ is infimum of $S \subseteq \mathbb{R}$ if and only if a_0 is a lower bound of S and if $\alpha > a_0$, then α is not a lower bound of S, i.e., there exists $x \in S$ such that $x < \alpha$.

The above two statements can be rephrased as follows:

- $b_0 \in \mathbb{R}$ is supremum of $S \subseteq \mathbb{R}$ if and only if b_0 is an upper bound of S and for every $\varepsilon > 0$, there exists $x \in S$ such that $b_0 - \varepsilon < x \le b_0$.

- $a_0 \in \mathbb{R}$ is infimum of $S \subseteq \mathbb{R}$ if and only if a_0 is a lower bound of S and for every $\varepsilon > 0$, there exists $x \in S$ such that $a_0 \le x < a_0 + \varepsilon$.

Exercise 1.14 Show that supremum (respectively, infimum) of a set $S \subseteq \mathbb{R}$, if exists, is unique. ◀

Exercise 1.15 Prove the following:

(i) If b_0 is the supremum of S, then there exists a sequence (x_n) in S which converges to b_0.

(ii) If a_0 is the infimum of S, then there exists a sqeuence (x_n) in S which converges to a_0. ◀

EXAMPLE 1.6 (i) If S is any of the intervals $(0,1)$, $[0,1)$, $(0,1]$, $[0,1]$, then 1 is the supremum of S and 0 is the infimum of S.

(ii) If $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then 1 is the supremum of S and 0 is the infimum of S.

(iii) For $k \in \mathbb{N}$, if $S_k = \{n \in \mathbb{N} : n \geq k\}$, then k is the infimum of S_k, and S_k has no supremum.

(iv) For $k \in \mathbb{N}$, if $S_k = \{n \in \mathbb{N} : n \leq k\}$, then k is the supremum of S_k, and S_k has no infimum. □

The above examples show the following:

- The supremum and/or infimium of a set S, if exists, need not belong to S.
- If S is not bounded above, then S does not have supremum.
- If S is not bounded below, then S does not have infimum.

However, we have the following properties for $\mathbb{R}$:

Least upper bound property: *If $S \subseteq \mathbb{R}$ is bounded above, then S has a least upper bound. We may write this least upper bound as $lub(S)$ or $\sup(S)$.*

Greatest lower bound property: *If $S \subseteq \mathbb{R}$ is bounded below, then S has a greatest lower bound*, and we write it as $glb(S)$ or $\inf(S)$.

Theorem 1.5 *(i) Every sequence which is monotonically increasing and bounded above is convergent.*

(ii) Every sequence which is monotonically decreasing and bounded below is convergent.

Proof. Suppose (a_n) is a monotonically increasing sequence of real numbers which is bounded above. Then the set $S := \{a_n : n \in \mathbb{N}\}$ is bounded above. Hence, by the least upper bound property of $\mathbb{R}$, S has a least upper bound, say b. Now, let $\varepsilon > 0$ be given. Then, by the definition of the least upper bound, there exists $N \in \mathbb{N}$ such that $a_N > b - \varepsilon$. Since $a_n \geq a_N$ for every $n \geq N$, we get

$$b - \varepsilon < a_N \leq a_n \leq b < b + \varepsilon \qquad \forall\, n \geq N.$$

Thus we have proved that $a_n \to b$ as $n \to \infty$.

To see the last part, suppose that (b_n) is a monotonically decreasing sequence which is bounded below. Then, it is seen that the sequence (a_n) defined by $a_n = -b_n$ for all $n \in \mathbb{N}$ is monotonically increasing and bounded above. Hence, by the first part of the theorem, $a_n \to a$ for some $a \in \mathbb{R}$. Then, $b_n \to b := -a$. ▮

Note that a convergent sequence need not be monotonically increasing or monotonically decreasing. For example, look at the sequence $((-1)^n/n)$.

1.1.3 Subsequences

Definition 1.11 A sequence (b_n) is called a **subsequence** of a sequence (a_n) if there is a strictly increasing sequence (k_n) of natural numbers such that $b_n = a_{k_n}$ for all $n \in \mathbb{N}$. □

Thus, subsequences of a real sequence (a_n) are of the form (a_{k_n}), where (k_n) is a strictly increasing sequence natural numbers.

For example, given a sequence (a_n), the sequences (a_{2n}), (a_{2n+1}), (a_{n^2}), (a_{2^n}) are some of its subsequences. As concrete examples, $(1/2n)$, and $(1/(2n+1))$, $(1/2^n)$ are subsequences of $(1/n)$.

A sequence may not converge, but it can have convergent subsequences. For example, we know that the sequence $((-1)^n)$ diverges, but the subsequences (a_n) and (b_n) defined by $a_n = 1, b_n = -1$ for all $n \in \mathbb{N}$ are convergent subsequences of $((-1)^n)$.

However, we have the following result.

Theorem 1.6 *If a sequence (a_n) converges to x, then all its subsequences converge to the same limit x.*

Exercise 1.16 Prove Theorem 1.6. ◀

What about the converse of the above theorem? Obviously, if all subsequences of a sequence (a_n) converge to the same limit x, then (a_n) also has to converge to x, as (a_n) is a subsequence of itself.

Suppose every subsequence of (a_n) has at least one subsequence which converges to x. Does the sequence (a_n) converges to x? The answer is affirmative, as the following theorem shows.

Theorem 1.7 *If every subsequence of (a_n) has at least one subsequence which converges to x, then (a_n) also converges to x.*

Proof. Proof is left as an exercise. ∎

We have seen in Theorem 1.3 that every convergent sequence is bounded, but a bounded sequence need not be convergent.

Question: For every bounded sequence, can we have a convergent subsequence?

The answer is in affirmative:

Theorem 1.8 (Bolzano-Weirstrass theorem). *Every bounded sequence of real numbers has a convergent subsequence.*

We shall relegate its proof to the appendix (See Section 1.4).

Remark 1.5 We may observe that the proof of Theorem 1.8 can be slightly modified so that we, in fact, have the following: *Every sequence in $\mathbb{R}$ has a monotonic subsequence.* ♦

Exercise 1.17 Prove the statement in italics in Remark 1.5. ◄

1.1.4 Further examples

EXAMPLE 1.7 Let a sequence (a_n) be defined as follows :

$$a_1 = 1, \quad a_{n+1} = \frac{2a_n + 3}{4}, \quad n = 1, 2, \ldots.$$

We show that (a_n) is monotonically increasing and bounded above.

Note that

$$a_{n+1} = \frac{2a_n + 3}{4} = \frac{a_n}{2} + \frac{3}{4} \geq a_n \quad \iff \quad a_n \leq \frac{3}{2}.$$

Thus it is enough to show that $a_n \leq 3/2$ for all $n \in \mathbb{N}$.

Clearly, $a_1 \leq 3/2$. If $a_n \leq 3/2$, then $a_{n+1} = a_n/2 + 3/4 < 3/4 + 3/4 = 3/2$. Thus, we have proved that $a_n \leq 3/2$ for all $n \in \mathbb{N}$. Hence, by Theorem 1.5, (a_n) converges. Let its limit be a. Then taking limit on both sides of $a_{n+1} = \frac{2a_n+3}{4}$ we have

$$a = \frac{2a + 3}{4} \quad \text{i.e.,} \quad 4a = 2a + 3 \quad \text{so that} \quad a = \frac{3}{2}.$$

□

EXAMPLE 1.8 Let a sequence (a_n) be defined as follows :

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \quad n = 1, 2, \ldots.$$

It is seen that, if the sequence converges, then its limit would be $\sqrt{2}$.

Since $a_1 = 2$, one may try to show that (a_n) is monotonically decreasing and bounded below.

Note that

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq a_n \quad \iff \quad a_n^2 \geq 2, \quad \text{i.e.,} \quad a_n \geq \sqrt{2}.$$

Clearly $a_1 \geq \sqrt{2}$. Now,

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \geq \sqrt{2} \qquad \Longleftrightarrow \qquad a_n^2 - 2\sqrt{2}a_n + 2 \geq 0,$$

i.e., if and only if $(a_n - \sqrt{2})^2 \geq 0$. This is true for every $n \in \mathbb{N}$. Thus, (a_n) is monotonically decreasing and bounded below, so that by Theorem 1.5, (a_n) converges.

□

EXAMPLE 1.9 Consider the sequence (a_n) with $a_n = (1+1/n)^n$ for all $n \in \mathbb{N}$. We show that (a_n) is monotonically increasing and bounded above. Hence, by Theorem 1.5, (a_n) converges.

Note that

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^n \\ &= 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\ldots 2.1}{n!}\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\ &\quad \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ &\leq a_{n+1}. \end{aligned}$$

Also

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \cdots + \frac{1}{n!}\frac{n(n-1)\ldots 2.1}{n^n} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &< 3. \end{aligned}$$

Thus, (a_n) is monotonically increasing bounded above. □

Exercise 1.18 Show that $(n^{1/n})_{n=3}^{\infty}$ is a monotonically decreasing sequence. ◀

In the four examples that follow, we shall be making use of Theorem 1.2 without mentioning it explicitly.

EXAMPLE 1.10 Consider the sequence (b_n) with

$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \qquad \forall\, n \in \mathbb{N}.$$

Clearly, (b_n) is monotonically increasing, and we have noticed in the last example that it is bounded above by 3. Hence, by Theorem 1.5, it converges.

Let (a_n) and (b_n) be as in Examples 1.9 and 1.10 respectively, and let a and b their limits. We show that $a = b$.

We have observed in last example that $2 \leq a_n \leq b_n \leq 3$. Hence, taking limits, it follows that $a \leq b$. Notice that

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\ &\quad \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Hence, for m, n with $m \leq n$, we have

$$\begin{aligned} a_n &\geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\ &\quad \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right). \end{aligned}$$

Taking limit as $n \to \infty$, we get (cf. Theorem 1.2 (c))

$$a \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} = b_m.$$

Now, taking limit as $m \to \infty$, we get $a \geq b$. Thus we have proved $a = b$.

The common limit in the above two examples is denoted by the letter e. □

EXAMPLE 1.11 Let $a > 0$. We show that, if $0 < a < 1$, then the sequence (a^n) converges to 0, and if $a > 1$, then (a^n) diverges to infinity.

(i) Suppose $0 < a < 1$. Then we can write $a = 1/(1+r)$, $r > 0$, and we have

$$a^n = 1/(1+r)^n = 1/\left(1 + nr + \cdots + r^n\right) < 1/(1+nr) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $a^n \to 0$ as $n \to \infty$.

An alternate way: Let $x_n = a^n$. Then (x_n) is monotonically decreasing and bounded below by 0. Hence (x_n) converges, to say x. Then $x_{n+1} = a^{n+1} = ax_n \to ax$. Hence, $x = ax$. This shows that $x = 0$.

(ii) Suppose $a > 1$. Then, since $0 < 1/a < 1$, the sequence $(1/a^n)$ converges to 0, so that (a^n) diverges to infinity. (*Why ?*) □

EXAMPLE 1.12 The sequence $(n^{1/n})$ converges and the limit is 1.

Note that $n^{1/n} = 1 + r_n$ for some sequence (r_n) of positive reals. Then we have

$$n = (1 + r_n)^n \geq \frac{n(n-1)}{2} r_n^2,$$

so that $r_n^2 \leq 2/(n-1)$ for all $n \geq 2$. Since $2/(n-1) \to 0$, by Theorem 1.2(c), that $r_n \to 0$, and hence by Theorem 1.2(c), $n^{1/n} = 1 + r_n \to 1$. □

Remark 1.6 In Example 1.12 for showing $n^{1/n} \to 1$ we did not require the fact that the limit is 1. But, if one is asked to show that $n^{1/n} \to 1$, then we can give another proof of the same by using only the definition, as follows:

Another proof without using Theorem 1.2. Let $\varepsilon > 0$ be given. To find $n_0 \in \mathbb{N}$ such that $n^{1/n} - 1 < \varepsilon$ for all $n \geq n_0$. Note that

$$\begin{aligned} n^{1/n} - 1 < \varepsilon \quad &\iff \quad n^{1/n} < 1 + \varepsilon \\ &\iff \quad n < (1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2}\varepsilon^2 + \ldots + \varepsilon^n. \end{aligned}$$

Hence

$$n^{1/n} - 1 < \varepsilon \quad \text{if} \quad n > 1 + 2/\varepsilon^2.$$

So, we may take any $n_0 \in \mathbb{N}$ which satisfies $n_0 \geq 2(1 + /\varepsilon^2)$. ♦

EXAMPLE 1.13 For any $a > 0$, $(a^{1/n})$ converges to 1.

If $a > 1$, then we can write $a^{1/n} = 1 + r_n$ for some sequence (r_n) of positive reals. Then we have

$$a = (1 + r_n)^n \geq n r_n \quad \text{so that} \quad r_n \leq a/n.$$

Since $a/n \to 0$, by Theorem 1.2(c), $r_n \to 0$, and hence by Theorem 1.2(c), $a^{1/n} = 1 + r_n \to 1$.

In case $0 < a < 1$, then $1/a > 1$. Hence, by the first part, $1/a^{1/n} = (1/a)^{1/n} \to 1$, so that $a^n \to 1$. □

Exercise 1.19 Give another proof for the result in Example 1.13 without using Theorem 1.2. ◀

EXAMPLE 1.14 Let (a_n) be a bounded sequence of non-negative real numbers. Then $(1 + a_n)^{1/n} \to 1$ as $n \to \infty$:

This is seen as follows: Let $M > 0$ be such that $0 \leq a_n \leq M$ for all $n \in \mathbb{N}$. Then,

$$1 \leq (1 + a_n)^{1/n} \leq (1 + M)^{1/n} \qquad \forall\, n \in \mathbb{N}.$$

By Example 1.13, $(1 + M)^{1/n} \to 0$. Hence the result follows by making use of part (d) of Theorem 1.2. □

EXAMPLE 1.15 As an application of some of the results discussed above, consider the sequence (a_n) with $a_n = (1+1/n)^{1/n}$, $n \in \mathbb{N}$. We already know that $\lim\limits_{n\to\infty} a_n = 1$. Now, another proof for the same:

Note that $1 \leq a_n \leq 2^{1/n}$, and $2^{1/n} \to 1$ as $n \to \infty$. □

EXAMPLE 1.16 Consider the sequence (a_n) with $a_n = (1 + n)^{1/n}$. Then $a_n \to 1$ as $n \to \infty$. We give two proofs for this result.

(i) Observe that $a_n = n^{1/n}\,(1 + 1/n)^{1/n}$. We already know that $n^{1/n} \to 1$, and $(1 + 1/n)^{1/n} \to 1$ as $n \to \infty$.

(ii) Observe that $n^{1/n} \le (1+n)^{1/n} \le (2n)^{1/n} = 2^{1/n}n^{1/n}$, where $n^{1/n} \to 1$ and $2^{1/n} \to 1$ as $n \to \infty$. $\square$

EXAMPLE 1.17 Suppose $a_n > 0$ for al $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell < 1$. We show that $a_n \to 0$.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell < 1$, there exists q such that $\ell < q < 1$ and $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \le q$ for all $n \ge N$. Hence,

$$0 \le a_n \le q^{n-N} a_N \qquad \forall\, n \ge N.$$

Now, since $q^{n-N} \to 0$ as $n \to \infty$, it follows that $a_n \to 0$ as $n \to \infty$. $\square$

Exercise 1.20 Suppose $a_n > 0$ for al $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell > 1$. Show that (a_n) diverges to ∞. ◀

EXAMPLE 1.18 Let $0 < a < 1$. Then $na^n \to 0$ as $n \to \infty$.

To see this let $a_n := na^n$ for $n \in \mathbb{N}$. Then we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)a^{n+1}}{na^n} = \frac{(n+1)a}{n} \qquad \forall\, n \in \mathbb{N}.$$

Hence, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a < 1$. Thus, the result follows from the last example. $\square$

Exercise 1.21 Obtain the result in Example 1.18 by using the arguments in Example 1.11. ◀

Remark 1.7 Suppose for each $k \in \mathbb{N}$, $a_n^{(k)} \to 0$, $b_n^{(k)} \to 1$ as $n \to \infty$, and also $a_n^{(n)} \to 0$, $b_n^{(n)} \to 1$ as $n \to \infty$. In view of Theorems 1.2 and 1.4, one may think that

$$a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(n)} \to 0 \quad \text{as} \quad n \to \infty$$

and

$$b_n^{(1)} b_n^{(2)} \cdots b_n^{(n)} \to 1 \quad \text{as} \quad n \to \infty.$$

Unfortunately, that is not the case. To see this consider

$$a_n^{(k)} = \frac{k}{n^2}, \qquad b_n^{(k)} = k^{1/n} \quad k, n \in \mathbb{N}.$$

Then, for each $k \in \mathbb{N}$, $a_n^{(k)} \to 0$, $b_n^{(k)} \to 1$ as $n \to \infty$, and also $a_n^{(n)} \to 0$, $b_n^{(n)} \to 1$ as $n \to \infty$. But,

$$a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(n)} = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \frac{n+1}{2n} \to \frac{1}{2} \quad \text{as} \quad n \to \infty$$

and

$$b_n^{(1)} b_n^{(2)} \cdots b_n^{(n)} = 1^{1/n} 2^{1/n} \cdots n^{1/n} = (n!)^{1/n} \not\to 1 \quad \text{as} \quad n \to \infty.$$

In fact, for any $k \in \mathbb{N}$, if $n \geq k$, then

$$(n!)^{1/n} \geq (k!)^{1/n}(k^{n-k})^{1/n} = (k!)^{1/n}k^{1-k/n} = k\Big(\frac{k!}{k^k}\Big)^{1/n}.$$

Since for any given $k \in \mathbb{N}$, $k\Big(\frac{k!}{k^k}\Big)^{1/n} \to k$ as $n \to \infty$, it follows that $(n!)^{1/n} \not\to 1$. In fact, there exists $n_0 \in \mathbb{N}$ such that

$$(n!)^{1/n} \geq \frac{k}{2} \qquad \forall\, n \geq \max\{n_0, k\}.$$

Thus, the sequence is $\big((n!)^{1/n}\big)$ is unbounded. ♦

EXAMPLE 1.19 Let $a_n = (n!)^{1/n^2}$, $n \in \mathbb{N}$. Then $a_n \to 1$ as $n \to \infty$. We give two proofs for this.

(i) Note that, for every $n \in \mathbb{N}$,

$$1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}.$$

Now, since $n^{1/n} \to 1$, we have $(n!)^{1/n^2} \to 1$.

(ii) By GM-AM inequality, for $n \in \mathbb{N}$,

$$(n!)^{1/n} = (1.2.\ldots.n)^{1/n} \leq \frac{1+2+\ldots+n}{n} = \frac{n+1}{2} \leq n.$$

Thus,

$$1 \leq (n!)^{1/n^2} \leq n^{1/n}.$$

Since $n^{1/n} \to 1$ we have $(n!)^{1/n^2} \to 1$. □

1.1.5 Cauchy sequence

Theorem 1.9 *If a real sequence (a_n) converges, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|a_n - a_m| < \varepsilon \quad \forall\, n, m \geq N.$$

Proof. Suppose $a_n \to a$ as $n \to \infty$, and let $\varepsilon > 0$ be given. Then we know that there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ for all $n \geq N$. Hence, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon \quad \forall\, n, m \geq N.$$

This completes the proof. ▮

Definition 1.12 A a sequence (a_n) is said to be a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall\, n, m \geq N.$$

□

Theorem 1.9 show that every convergent sequence is a Cauchy sequence. In particular, if (a_n) is not a Cauchy sequence, then (a_n) does not converge to any $a \in \mathbb{R}$. Thus, Theorem 1.9 may help us to show that certain sequence is not convergent. For example, see the follwing exercise.

Exercise 1.22 Let $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ for $n \in \mathbb{N}$. Then show that (s_n) is not a Cauchy sequence. [*Hint*: For any $n \in \mathbb{N}$, note that $s_{2n} - s_n \geq \frac{1}{2}$.] ◀

Let us observe certain properties of Cauchy sequences.

Theorem 1.10 *Let (a_n) be a Cauchy sequence. Then we have the following.*

(i) (a_n) is a bounded sequence.

(ii) If (a_n) has a convergent subsequence with limit a, then the sequence (a_n) itself will converge to a.

Proof. Let (a_n) be Cauchy sequence and $\varepsilon > 0$ be given.

(i) Let $N \in \mathbb{N}$ be such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$. In particular,

$$|a_n| \leq |a_n - a_N| + |a_N| \leq \varepsilon + |a_N| \qquad \forall\, n \geq N.$$

This proves (i).

(ii) Suppose (a_{n_k}) is a convergent subsequence of (a_n), and let $a = \lim\limits_{k\to\infty} a_{n_k}$. Let $n_0 \in \mathbb{N}$ be such that $|a_{n_k} - a| < \varepsilon$ for all $k \geq n_0$. Hence, we have

$$|a_k - a| \leq |a_k - a_{n_k}| + |a_{n_k} - a| < 2\varepsilon \qquad \forall\ k \geq \max\{N, n_0\}.$$

Thus, $a_n \to a$ as $n \to \infty$. ∎

In fact, the converse of Theorem 1.9 also holds:

Theorem 1.11 *Every Cauchy sequence of real numbers converges.*

Proof. Let (a_n) be a Cauchy sequence. By Theorem 1.10(i), (a_n) is bounded. Then, by Bolzano-Weierstrass theorem (Theorem 1.8), (a_n) has a convergent subsequence (a_{n_k}). Let $a = \lim\limits_{k\to\infty} a_{n_k}$. Now, by Theorem 1.10(ii), $a_n \to a$ as $n \to \infty$. ∎

Remark 1.8 For an alternate proof of Theorem 1.11, without using Bolzano-Weierstrass theorem, see Section 1.4. ♦

Here is an example of a general nature.

EXAMPLE 1.20 Let (a_n) be a sequence of real numbers. Suppose there exists a positive real number $\rho < 1$ such that

$$|a_{n+1} - a_n| \leq \rho |a_n - a_{n-1}| \quad \forall\ n \in \mathbb{N}, n \geq 2.$$

Then (a_n) is a cauchy sequence. To see this first we observe that

$$|a_{n+1} - a_n| \leq \rho^{n-1}|a_2 - a_1| \quad \forall\, n \in \mathbb{N}, n \geq 2.$$

Hence, for $n > m$,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - a_{n-1}| + \ldots + |a_{m+1} - a_m| \\ &\leq (\rho^{n-2} + \ldots + \rho^{m-1})|a_2 - a_1| \\ &\leq \rho^{m-1}(1 + \rho + \ldots + \rho^{n-m-3})|a_2 - a_1| \\ &\leq \frac{\rho^{m-1}}{1-\rho}|a_2 - a_1|. \end{aligned}$$

Since $\rho^{m-1} \to 0$ as $m \to \infty$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$. □

Exercise 1.23 Given $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let (a_n) be a sequence of real numbers defined by $a_1 = a, a_2 = b$ and

$$a_{n+1} = (1+\lambda)a_n - \lambda a_{n-1} \quad \forall\, n \in \mathbb{N},\ n \geq 2.$$

Show that (a_n) is a Cauchy sequence and its limit is $(b + \lambda a)/(1 - \lambda)$. ◀

Exercise 1.24 Suppose f is a function defined on an interval J. If there exists $0 < \rho < 1$ such that

$$|f(x) - f(y)| \leq \rho|x - y| \quad \forall\, x, y \in J,$$

then for any $a \in J$, the sequence (a_n) defined by

$$a_1 = f(a), \quad a_{n+1} := f(a_n) \quad \forall\, n \in \mathbb{N},$$

is a Cauchy sequence. Show also that the limit of the sequence (a_n) is independent of the choice of a. ◀

1.2 Series of Real Numbers

Definition 1.13 A **series** of real numbers is an expression of the form

$$a_1 + a_2 + a_3 + \ldots,$$

or more compactly as $\sum_{n=1}^{\infty} a_n$, where (a_n) is a sequence of real numbers.

The number a_n is called the n-th term of the series and the sequence $s_n := \sum_{i=1}^{n} a_i$ is called the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$.

□

1.2.1 Convergence and divergence of series

Definition 1.14 A series $\sum_{n=1}^{\infty} a_n$ is said to *converge* (to $s \in \mathbb{R}$) if the sequence $\{s_n\}$ of partial sums of the series converge (to $s \in \mathbb{R}$).

If $\sum_{n=1}^{\infty} a_n$ converges to s, then we write $\sum_{n=1}^{\infty} a_n = s$.

A series which does not converge is called a *divergent series.* □

A necessary condition

Theorem 1.12 *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Converse does not hold.*

Proof. Clearly, if s_n is the n-th partial sum of the convergent series $\sum_{n=1}^{\infty} a_n$, then

$$a_n = s_n - s_{n-1} \to 0 \quad \text{as} \quad n \to \infty.$$

To see that the converse does not hold it is enough to observe that the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ diverges whereas $a_n \to 0$. ▮

The proof of the following corollary is immediate from the above theorem.

Corollary 1.13 *Suppose (a_n) is a sequence of positive terms such that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

The above theorem and corollary shows, for example, that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

EXAMPLE 1.21 We have seen that the sequence (s_n) with $s_n = \sum_{k=1}^{n} \frac{1}{k!}$ converges. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. Also, we have seen that the sequence (σ_n) with $\sigma_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

EXAMPLE 1.22 Consider the geometric series $\sum_{n=1}^{\infty} aq^{n-1}$, where $a, q \in \mathbb{R}$. Note that $s_n = a + aq + \ldots + aq^{n-1}$ for $n \in \mathbb{N}$. Clearly, if $a = 0$, then $s_n = 0$ for all $n \in \mathbb{N}$. Hence, assume that $a \neq 0$. Then we have

$$s_n = \begin{cases} na & \text{if} \quad q = 1, \\ \frac{a(1-q^n)}{1-q} & \text{if} \quad q \neq 1. \end{cases}$$

Thus, if $q = 1$, then (s_n) is not bounded; hence not convergent. If $q = -1$, then we have

$$s_n = \begin{cases} a & \text{if} \quad n \text{ odd}, \\ 0 & \text{if} \quad n \text{ even}. \end{cases}$$

Thus, (s_n) diverges for $q = -1$ as well. Now, assume that $|q| \neq 1$. In this case, we have

$$\left| s_n - \frac{a}{1-q} \right| = \frac{|a|}{|1-q|} |q|^n.$$

This shows that, if $|q| < 1$, then (s_n) converges to $\frac{a}{1-q}$, and if $|q| > 1$, then (s_n) is not bounded, hence diverges. □

Theorem 1.14 *Suppose (a_n) and (b_n) are sequences such that for some $k \in \mathbb{N}$, $a_n = b_n$ for all $n \geq k$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.*

Proof. Suppose s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Let $\alpha = \sum_{i=1}^{k} a_i$ and $\beta = \sum_{i=1}^{k} b_i$. Then we have

$$s_n - \alpha = \sum_{i=k+1}^{n} a_i = \sum_{i=k+1}^{n} b_i = \sigma_n - \beta \quad \forall\, n \geq k.$$

From this it follows that the sequence (s_n) converges if and only if (σ_n) converges. ∎

From the above theorem it follows if $\sum_{n=1}^{\infty} b_n$ is obtained from $\sum_{n=1}^{\infty} a_n$ by omitting or adding a finite number of terms, then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} b_n \quad \text{converges.}$$

In particular,

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} a_{n+k} \quad \text{converges}$$

for any $k \in \mathbb{N}$.

The proof of the following theorem is left as an exercise.

Theorem 1.15 *Suppose $\sum_{n=1}^{\infty} a_n$ converges to s and $\sum_{n=1}^{\infty} b_n$ converges to σ. Then for every $\alpha, \beta \in \mathbb{R}$, $\sum_{n=1}^{\infty}(\alpha\, a_n + \beta\, b_n)$ converges to $\alpha\, s + \beta\, \sigma$.*

1.2.2 Some tests for convergence

Theorem 1.16 (**Comparison test**) *Suppose (a_n) and (b_n) are sequences of* **non-negative terms**, *and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then,*

(i) $\sum_{n=1}^{\infty} b_n$ *converges* $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ *converges,*

(ii) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Longrightarrow$ $\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. Suppose s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. By the assumption, we get $0 \leq s_n \leq \sigma_n$ for all $n \in \mathbb{N}$, and both (s_n) and (σ_n) are monotonically increasing.

(i) Since (σ_n) converges, it is bounded. Let $M > 0$ be such that $\sigma_n \leq M$ for all $n \in \mathbb{N}$. Then we have $s_n \leq M$ for all $n \in \mathbb{N}$. Since (s_n) are monotonically increasing, it follows that (s_n) converges.

(ii) Proof of this part follows from (i) (*How?*). ∎

Corollary 1.17 *Suppose (a_n) and (b_n) are sequences of* **positive terms**.

(a) Suppose $\ell := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists. Then we have the following:

(i) If $\ell > 0$, then $\sum_{n=1}^{\infty} b_n$ converges $\iff$ $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\ell = 0$, then $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ converges.

Proof. (a) Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$.

(i) Let $\ell > 0$. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\ell - \varepsilon < \frac{a_n}{b_n} < \ell + \varepsilon$ for all $n \geq N$. Equivalently, $(\ell - \varepsilon)b_n < a_n < (\ell + \varepsilon)b_n$ for all $n \geq N$. Had we taken $\varepsilon = \ell/2$, we would get $\frac{\ell}{2}b_n < a_n < \frac{3\ell}{2}b_n$ for all $n \geq N$. Hence, the result follows by comparison test.

(ii) Suppose $\ell = 0$. Then for $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $-\varepsilon < \frac{a_n}{b_n} < \varepsilon$ for all $n \geq N$. In particular, $a_n < \varepsilon b_n$ for all $n \geq N$. Hence, we get the result by using comparison test.

(b) By assumption, there exists $N \in \mathbb{N}$ such that $\frac{a_n}{b_n} \geq 1$ for all $n \geq N$. Hence the result follows by comparison test. ∎

EXAMPLE 1.23 We have already seen that the sequence (s_n) with $s_n = \sum_{k=1}^{n} \frac{1}{k!}$ converges. Here is another proof for the same fact: Note that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges, it follows from the above theorem that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. □

EXAMPLE 1.24 Since $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for all $n \in \mathbb{N}$, and since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows from the above theorem that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. □

Theorem 1.18 (De'Alembert's ratio test) *Suppose (a_n) is a sequence of positive terms such that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ exists. Then we have the following:*

(i) If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Suppose $\ell < q < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} < q \quad \forall\, n \geq N.$$

In particular,

$$a_{n+1} < q\, a_n < q^2 a_{n-1} < \ldots < q^n\, a_1, \forall\, n \geq N.$$

Since $\sum_{n=1}^{\infty} q^n$ converges, by comparison test, $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) Let $1 < p < \ell$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} > p > 1 \, n \geq N.$$

From this it follows that (a_n) does not converge to 0. Hence $\sum_{n=1}^{\infty} a_n$ diverges. ∎

Theorem 1.19 (**Cauchy's root test**) *Suppose (a_n) is a sequence of positive terms such that $\lim\limits_{n\to\infty} {a_n}^{1/n} = \ell$ exists. Then we have the following:*

(i) If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Suppose $\ell < q < 1$. Then there exists $N \in \mathbb{N}$ such that

$${a_n}^{1/n} < q \quad \forall\, n \geq N.$$

Hence, $a_n < q^n$ for all $n \geq N$. Since the $\sum_{n=1}^{\infty} q^n$ converges, by comparison test, $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) Let $1 < p < \ell$. Then there exists $N \in \mathbb{N}$ such that

$${a_n}^{1/n} > p > 1 \quad \forall\, n \geq N.$$

Hence, $a_n \geq 1$ for all $n \geq N$. Thus, (a_n) does not converge to 0. Hence, $\sum_{n=1}^{\infty} a_n$ also diverges. ∎

Remark 1.9 We remark that both d'Alembert's test and Cauchy test are silent for the case $\ell = 1$. But, for such case, we may be able to infer the convergence or divergence by some other means. ♦

EXAMPLE 1.25 For every $x \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges:

Here, $a_n = \frac{x^n}{n!}$. Hence

$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \quad \forall\, n \in \mathbb{N}.$$

Hence, it follows that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, so that by d'Alemberts test, the series converges. □

EXAMPLE 1.26 The series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges: Here

$${a_n}^{1/n} = \frac{n}{2n+1} \to \frac{1}{2} < 1.$$

Hence, by Cauchy's test, the series converges. □

EXAMPLE 1.27 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. In this series, we see that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim_{n\to\infty} {a_n}^{1/n}$. However, the n-th partial sum s_n is given by

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}.$$

Hence $\{s_n\}$ converges to 1. □

EXAMPLE 1.28 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. In this case, we see that

$$\frac{1}{(n+1)^2} \le \frac{1}{n(n+1)} \quad \forall\, n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, by comparison test, the given series also converges. □

EXAMPLE 1.29 Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison test, we see that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p \ge 2$ and diverges for $p \le 1$. □

EXAMPLE 1.30 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$. To discuss this example, consider the function $f(x) := 1/x^p$, $x \ge 1$. Then, using teh fact that indefinite integral of x^q for $q \ne -1$ is $x^{q+1}/(q+1)$, we see that for each $k \in \mathbb{N}$,

$$k - 1 \le x \le k \Longrightarrow \frac{1}{k^p} \le \frac{1}{x^p} \Longrightarrow \frac{1}{k^p} \le \int_{k-1}^{k} \frac{dx}{x^p}.$$

Hence,

$$\sum_{k=2}^{n} \frac{1}{k^p} \le \sum_{k=2}^{n} \int_{k-1}^{k} \frac{dx}{x^p} = \int_{1}^{n} \frac{dx}{x^p} = \frac{n^{1-p} - 1}{1-p} \le \frac{1}{p-1}.$$

Thus,

$$s_n := \sum_{k=1}^{n} \frac{1}{k^p} \le \frac{1}{p-1} + 1.$$

Hence, (s_n) is monotonically increasing and bounded above. Therefore, (s_n) converges. □

A more general result on convergence of series in terms of integrals will be proved in Chapter 3.

1.2.3 Alternating series

Definition 1.15 A series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ where (u_n) is a sequence of positive terms is called an **alternating series**. □

Theorem 1.20 (**Leibniz's theorem**) *Suppose (u_n) is a sequence of positive terms such that $u_n \ge u_{n+1}$ for all $n \in \mathbb{N}$, and $u_n \to 0$ as $n \to \infty$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.*

Proof. Let s_n be the n-th partial sum of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$. We observe that

$$s_{2n+1} = s_{2n} + u_{2n+1} \quad \forall\ n \in \mathbb{N}.$$

Since $u_n \to 0$ as $n \to \infty$, it is enough to show that (s_{2n}) converges (Why?). Note that

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}),$$

$$s_{2n} = u_1 - (u_2 - u_3) - \ldots (u_{2n-2} - u_{2n-1}) - u_{2n}$$

for all $n \in \mathbb{N}$. Since $u_i - u_{i+1} \geq 0$ for each $i \in \mathbb{N}$, (s_{2n}) is monotonically increasing and bounded above. Therefore (s_{2n}) converges. In fact, if $s_{2n} \to s$, then we have $s_{2n+1} = s_{2n} + u_{2n+1} \to s$, and hence $s_n \to s$ as $n \to \infty$. ∎

By the above theorem the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Likewise, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$ also converge[1]

Suppose (u_n) is as in Leibniz's theorem (Theorem 1.20), and let $s \in \mathbb{R}$ be such that $s_n \to s$, where s_n is the n^{th} partial sum of $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n$.

How fast (s_n) **converges to** s**?**

In the proof of Theorem 1.20, we have shown that $\{s_{2n}\}$ is a monotonically increasing sequence. Similarly, it can be shown that $\{s_{2n-1}\}$ is a monotonically decreasing sequence.

Since (s_{2n-1}) is monotonically decreasing and (s_{2n}) is monotonically increasing, we have

$$s_{2n-1} = s_{2n} + u_{2n} \leq s + u_{2n}, \qquad s \leq s_{2n+1} = s_{2n} + u_{2n+1}.$$

Thus,

$$s_{2n-1} - s \leq u_{2n}, \qquad s - s_{2n} \leq u_{2n+1}.$$

Consequently,

$$|s - s_n| \leq u_{n+1} \quad \forall\ n \in \mathbb{N}.$$

1.2.4 Absolute convergence

Definition 1.16 A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely**, if $\sum_{n=1}^{\infty} |a_n|$ converges. □

Theorem 1.21 *Every absolutely convergent series converges.*

[1]The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ appear in the work of a Kerala mathematician *Madhava* did around 1425 which wa presented later in the year around 1550 by another Kerala mathematician *Nilakantha*. The discovery of the above series is normally attributed to Leibniz and James Gregory after nearly 300 years of its discovery.

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively. Then, for $n > m$, we have

$$|s_n - s_m| = \Big| \sum_{j=m+1}^{n} a_n \Big| \le \sum_{j=m+1}^{n} |a_n| = |\sigma_n - \sigma_m|.$$

Since, $\{\sigma_n\}$ converges, it is a Cauchy sequence. Hence, form the above relation it follows that $\{s_n\}$ is also a Cauchy sequence. Therefore, by the *Cauchy criterion*, it converges. ∎

Another proof without using Cauchy criterion. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respectively. Then it follows that

$$s_n + \sigma_n = 2p_n,$$

where p_n is the sum of all positive terms from $\{a_1, \ldots, a_n\}$. Since $\{\sigma_n\}$ converges, it is bounded, and since $p_n \le \sigma_n$ for all $n \in \mathbb{N}$, the sequence $\{p_n\}$ is also bounded. Moreover, $\{p_n\}$ is monotonically increasing. Hence $\{p_n\}$ converge as well. Thus, both $\{\sigma_n\}$, $\{p_n\}$ converge. Now, since $s_n = 2p_n - \sigma_n$ for all $n \in \mathbb{N}$, the sequence $\{s_n\}$ also converges. ∎

Definition 1.17 A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if $\sum_{n=1}^{\infty} a_n$ converges, but not absolutely. □

EXAMPLE 1.31 We observe the following:

(i) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

(ii) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

(iii) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent. □

EXAMPLE 1.32 For any $\alpha \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2}$ is absolutely convergent: Note that

$$\left| \frac{\sin(n\alpha)}{n^2} \right| \le \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, $\sum_{n=1}^{\infty} \left| \frac{\sin(n\alpha)}{n^2} \right|$ also converges. □

EXAMPLE 1.33 The series $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \log n}{n \log(\log n)}$ is conditionally convergent. To see this, let $u_n = \dfrac{\log n}{n \log(\log n)}$. Since $n \ge \log n \ge \log(\log n)$ we have

$$\frac{1}{n} \le \frac{\log n}{n \log(\log n)} \le \frac{1}{\log(\log n)} \qquad (*)$$

so that $u_n \to 0$. It can be easily seen that $u_{n+1} \leq u_n$. Hence, by Leibnitz theorem, the given series converges. Inequality $(*)$ also shows that the series $\sum_{n=3}^\infty u_n$ does not converge. $\square$

Here are two more results whose proofs are based on some advanced topics in *analysis*

Theorem 1.22 *Suppose $\sum_{n=1}^\infty a_n$ is an absolutely convergent series and (b_n) is a sequence obtained by rearranging the terms of (a_n). Then $\sum_{n=1}^\infty b_n$ is also absolutely convergent, and $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty b_n$.*

Theorem 1.23 *Suppose $\sum_{n=1}^\infty a_n$ is a conditionally convergent series. The for every $\alpha \in \mathbb{R}$, there exists a sequence (b_n) whose terms are obtained by rearranging the terms of (a_n) such that $\sum_{n=1}^\infty b_n = \alpha$.*

To illustrate the last theorem consider the conditionally convergent series $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$. Consider the following rearrangement of this series:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} + \cdots .$$

Thus, if $a_n = \frac{(-1)^{n+1}}{n}$ for all $n \in \mathbb{N}$, the rearranged series is $\sum_{n=1}^\infty b_n$, where

$$b_{3k-2} = \frac{1}{2k-1}, \qquad b_{3k-1} = \frac{1}{4k-2}, \qquad b_{3k} = \frac{1}{4k}$$

for $k = 1, 2, \ldots$. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ respectively. Then we see that

$$\begin{aligned}
\sigma_{3k} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k} \\
&= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}\right) \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{4k-2} - \frac{1}{4k}\right) \\
&= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right)\right] \\
&= \frac{1}{2} s_{2k}.
\end{aligned}$$

Also, we have

$$\sigma_{3k+1} = \sigma_{3k} + \frac{1}{2k+1}, \qquad \sigma_{3k+2} = \sigma_{3k} + \frac{1}{2k+1} - \frac{1}{4k+2}.$$

We know that $\{s_n\}$ converge. Let $\lim_{n\to\infty} s_n = s$. Since, $a_n \to 0$ as $n \to \infty$, it then follows that

$$\lim_{k\to\infty} \sigma_{3k} = \frac{s}{2}, \qquad \lim_{k\to\infty} \sigma_{3k+1} = \frac{s}{2}, \qquad \lim_{k\to\infty} \sigma_{3k+2} = \frac{s}{2}.$$

Hence, we can infer that $\sigma_n \to s/2$ as $n \to \infty$.

1.3 Additional exercises

1.3.1 Sequences

1. Prove that $x_n \longrightarrow l$ as $n \to \infty$ if and only if $|x_n - l| \to 0$ as $n \longrightarrow \infty$, and in that case $|x_n| - |l| \longrightarrow 0$ as $n \to \infty$.

2. In each of the following, establish the convergence or divergence of the sequence (a_n), where a_n is:

$$(i)\ \frac{n}{n+1} \qquad (ii)\ \frac{(-1)^n n}{n+1} \qquad (iii)\ \frac{2n}{3n^2+1}, \qquad (iv)\ \frac{2n^2+3}{3n^2+1}.$$

3. Suppose (a_n) is a real sequence such that $a_n \to 0$ as $n \to \infty$. Show the following:

   (a) The sequence (a_n^2) converges to 0.

   (b) If $a_n > 0$ for all n, then the sequence $(1/a_n)$ diverges to infinity.

4. Let $\{x_n\}$ be a sequence defined recursively by $x_{n+2} = x_{n+1} + x_n,\ n \geq 1$ with $x_1 = x_2 = 1$. Show that $\{x_n\}$ diverges to ∞.

5. Let $x_n = \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $\{x_n\}$, $\{nx_n\}$ and $\{\sqrt{n}x_n\}$ are convergent. Find their limits.

6. Let $x_n = \sum_{k=1}^{n} \frac{1}{n+k},\ n \in \mathbb{N}$. Show that $\{x_n\}$ is convergent.

7. If (a_n) converges to x and $a_n \geq 0$ for all $n \in \mathbb{N}$, then show that $x \geq 0$ and $(\sqrt{a_n})$ converges to $\sqrt{x}$.

8. Prove the following:

   (a) If $\{x_n\}$ is increasing and unbounded, then $x_n \longrightarrow +\infty$ as $n \longrightarrow \infty$.

   (b) If $\{x_n\}$ is decreasing and unbounded, prove that $x_n \longrightarrow -\infty$ as $n \longrightarrow \infty$.

9. Let $a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n}$ for all $n \in \mathbb{N}$. Show that (a_n) converges. Also, find its limit.

10. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{4}(2a_n + 3)$ for all $n \in \mathbb{N}$. Show that $\{a_n\}$ is monotonically increasing and bounded above. Find its limit.

11. Let $a_1 = 1$ and $a_{n+1} = \dfrac{a_n}{1+a_n}$ for all $n \in \mathbb{N}$. Show that $\{a_n\}$ converges. Find its limit.

12. Prove that if (a_n) is a Cauchy sequence having a subsequence which converges to a, then (a_n) itself converges to a.

13. Suppose (a_n) is a sequence such that the subsequences (a_{2n-1}) and (a_{2n}) converge to the same limit, say a. Show that (a_n) also converges to a.

14. Let $\{x_n\}$ be a monotonically increasing sequence such that $\{x_{3n}\}$ is bounded. Is $\{x_n\}$ convergent? Justify your answer.

15. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{4}(2a_n + 3)$ for all $n \in \mathbb{N}$. Show that (a_n) is monotonically increasing and bounded above. Find its limit.

16. Give an example in support of the statement: If a sequence $\{a_n\}$ has the property that $a_{n+1} - a_n \to 0$ as $n \to \infty$, then $\{a_n\}$ need not converge.

17. Let $a > 0$ and $x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}$, $n \in \mathbb{N}$. Discuss the convergence of the sequence (x_n).

18. If $0 < a < b$ and $a_n = (a^n + b^n)^{1/n}$ for all $n \in \mathbb{N}$, then show that (a_n) converges to b. [*Hint:* Note that $(a^n + b^n)^{1/n} = b(1 + (\frac{a}{b})^n)^{1/n}$.]

19. If $0 < a < b$ and $a_{n+1} = (a_n b_n)^{1/2}$ and $b_{n+1} = \frac{a_n + b_n}{2}$ for all $n \in \mathbb{N}$ with $a_1 = a$, $b_1 = b$, then show that (a_n) and (b_n) converge to the same limit. [*Hint:* First observe that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all $n \in \mathbb{N}$.]

20. Let $a_1 = 1/2$ and $b = 1$, and let $a_{n+1} = (a_n b_n)^{1/2}$ and $b_{n+1} = \frac{2a_n b_n}{a_n + b_n}$ for all $n \in \mathbb{N}$. Show that (a_n) and (b_n) converge to the same limit. [*Hint:* First observe that $b_n \le b_{n+1} \le a_{n+1} \le a_n$ for all $n \in \mathbb{N}$.]

21. Let $a_1 = 1/2,\ b = 1,\ a_n = (a_{n-1} b_{n-1})^{1/2}$ and $\frac{1}{b_n} = \frac{1}{2}(\frac{1}{a_n} + \frac{1}{b_{n-1}})$. Prove that $a_{n-1} < a_n < b_n < b_{n-1}$ for all $n \in \mathbb{N}$. Deduce that both the sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit α, $1/2 < \alpha < 1$.

1.3.2 Series

(These problems were prepared for the students of MA1010, Aug-Nov, 2010.)

1. Using partial fractions, prove that $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = 1$

2. Test the following series for convergence:

$(a)\ \displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ $\quad (b)\ \displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n$ $\quad (c)\ \displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$(d)\ \displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} 4^n$ $\quad (e)\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n}\right)$ $\quad (f)\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{\sqrt{n}}$

$(g)\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 3}$ $\quad (h)\ \displaystyle\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} + \ldots$

$(i)\ \displaystyle\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{20}} + \frac{1}{\sqrt{30}} + \ldots + \frac{1}{\sqrt{10n}} + \ldots$ $\quad (j)\ \displaystyle 2 + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{n+1}{n} + \ldots$

(k) $\dfrac{1}{\sqrt[3]{7}}+\dfrac{1}{\sqrt[3]{8}}+\dfrac{1}{\sqrt[3]{9}}+...+\dfrac{1}{\sqrt[3]{n+6}}+...$

(l) $\dfrac{1}{2}+\left(\dfrac{2}{3}\right)^4+\left(\dfrac{3}{4}\right)^9+...+\left(\dfrac{n}{n+1}\right)^{n^2}+...$ $\quad(m)$ $\dfrac{1}{2}+\dfrac{2}{5}+\dfrac{3}{10}+...+\dfrac{n}{n^2+1}+...$

3. Is the Leibniz Theorem applicable to the series:
$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+...+\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}+...$$
Does the above series converge?

   Justify your answer.

4. Find out whether (or not) the following series converge absolutely or conditionally:

   (a) $1-\dfrac{1}{3^2}+\dfrac{1}{5^2}-\dfrac{1}{7^2}+\dfrac{1}{9^2}+...+(-1)^{n+1}\dfrac{1}{(2n-1)^2}+...$

   (b) $\dfrac{1}{\ln 2}-\dfrac{1}{\ln 3}+\dfrac{1}{\ln 4}-\dfrac{1}{\ln 5}+...+(-1)^n\dfrac{1}{\ln n}+...$

5. Find the sum of the series: $\dfrac{1}{1.2.3}+\dfrac{1}{2.3.4}+...+\dfrac{1}{n.(n+1).(n+2)}+...$

6. Test for the convergence of the following series:

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)\sqrt{n}}$ $\quad(b)$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ $\quad(c)$ $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(a+n)^p(b+n)^p},\quad a,b,p,q>0$ $\quad(e)$ $\displaystyle\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

   (f) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$ $\quad(g)$ $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n!}}{n!}$ $\quad(h)$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

   (i) $\displaystyle\sum_{n=1}^{\infty}(n+6)^{-1/3}$ $\quad(j)$ $\displaystyle\sum_{n=1}^{\infty}(\log n)^{-n}$ $\quad(k)$ $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

   (l) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{nx}{1+n}\right)^n$ $\quad(m)$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}$ $\quad(n)$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n(n+1)(n+2)}}$

   (o) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ $\quad(p)$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$

7. Examine the following series for absolute / conditional convergence:

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}$ $\quad(b)$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{3^{n-1}}$ $\quad(c)$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n(\log n)^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\log n}{n\log\log n}$ $\quad(e)$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ $\quad(f)$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}$

8. Let (a_n) be a sequence of non-negative numbers and (a_{k_n}) be a subsequence (a_n). Show that, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_{k_n}$ also converges. Is the converse true? Why?

1.4 Appendix

Proof of Theorem 1.8 (Bolzano-Weirstrass theorem). [2] Let (a_n) be a bounded sequence in $\mathbb{R}$. For each $k \in \mathbb{N}$, consider the set

$$E_k := \{a_n : n \geq k\},$$

and let $b_k := \sup E_k$, $k \in \mathbb{N}$. Clearly, $b_1 \geq b_2 \geq \ldots$.

We consider the following two mutually exclusive cases:

(i) For every $k \in \mathbb{N}$, $b_k \in E_k$

(ii) There exists $k \in \mathbb{N}$ such that $b_k \notin E_k$.

In case (i), (b_k) is a bounded monotonically decreasing subsequence of (a_n), and hence, by Theorem 1.5(ii), (b_k) converges.

Now, suppose that case (ii) holds, and let $k \in \mathbb{N}$ such that $b_k \notin E_k$. Then for every $n \geq k$, there exists $n' > n$ such that $a_{n'} > a_n$. Take $n_1 = k$ and $n_2 = n'$. Then we have $n_1 < n_2$ and $a_{n_1} < a_{n_2}$. Again, since $n_2 \in E_k$, there exists $n_3 > n_2$ such that $a_{n_2} < a_{n_3}$. Continuing this way, we obtain a monotonically increasing subsequence $(a_{n_j})_{j=1}^{\infty}$ of (a_n). Again, since (a_n) is bounded, by Theorem **??**by Theorem 1.5(ii)), $(a_{n_j})_{j=1}^{\infty}$ converges. ∎

An alternate proof for Theorem 1.11. Let (x_n) be Cauchy sequence, and $\varepsilon > 0$ be given. Then there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_\varepsilon$. In particular, $|x_n - x_{n_\varepsilon}| < \varepsilon$ for all $n \geq n_\varepsilon$. Hence, for any $\varepsilon_1, \varepsilon_2 > 0$,

$$x_{n_{\varepsilon_1}} - \varepsilon_1 < x_n < x_{n_{\varepsilon_2}} + \varepsilon_2 \qquad \forall\, n \geq \max\{n_{\varepsilon_1}, n_{\varepsilon_2}\}. \tag{$*$}$$

From this, we see that the set $\{x_{n_\varepsilon} - \varepsilon : \varepsilon > 0\}$ is bounded above, and

$$x := \sup\{x_{n_\varepsilon} - \varepsilon : \varepsilon > 0\}$$

satisfies

$$x_{n_\varepsilon} - \varepsilon < x < x_{n_\varepsilon} + \varepsilon \qquad \forall\, n \geq n_\varepsilon. \tag{$**$}$$

From $(*)$ and $(**)$, we obtain

$$|x_n - x| < \varepsilon \qquad \forall\, n \geq n_\varepsilon.$$

This completes the proof. ∎

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[2] From the book: *Mathematical Analysis: a straight forward approach* by K.G. Binmore.